Elements of the Green Ring related to Alperin's Weight Conjecture

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I am honoured and grateful to be able to speak at this conference in honour of my colleague and friend Michel Broué, and am sorry I can not be there in person, since I would have liked to be able to return to China for this occasion. I first met Michel Broué in 1981 in Chicago. I have learned much from Michel Broué over the decades (including the fact that I first heard about subgroups complexes from Michel in 1981, when he spoke about some work of S. Bouc!).

We first state Alperin's weight conjecture and its blockwise refinement (after choosing a p-modular system $(R, \mathbb{K}, \mathbb{F})$.

Alperin's Weight Conjecture: *We have*

$$\ell(G) = \sum_{Q \in \mathcal{Q}} f_0(N_G(Q)/Q).$$

Blockwise Refinement: *For each block B of RG, we have*

$$\ell(B) = \sum_{Q \in \mathcal{Q}} f_0(B(Q)).$$

Here, $\ell(G)$ is the number of simple modules for G and $\ell(B)$ is the number of simple modules in the block B. Here Q runs over p-subgroups of G up to conjugacy, $f_0(G)$ is the number of projective simple $\mathbb{F}G$-modules, and $f_0(B(Q))$ denotes the number of projective simple $\mathbb{F}N_G(Q)/Q$-modules which lie in Brauer correspondents of B for $N_G(Q)$.

Definition: The *Truncated Conjugation Module for G* is the RG-module, denoted by $\mathsf{T}_{c,p,G}$, which is the lift to RG-module of the projective $\mathbb{F}G$-module $\oplus_S \mathrm{Hom}_{\mathbb{F}}(S, P(S))$, where S runs over the isomorphism types of simple $\mathbb{F}G$-modules.

The projective RG-module $T_{c,p,G}$ affords a character $\Lambda_{c,p,G}$, which vanishes on all p-singular elements of G, and takes value $|C_G(x)|$ at each p-regular $x \in G$.

Note that the projective cover of the trivial RG-module occurs with multiplicity $\ell(G)$ as a summand of $\mathsf{T}_{c,p,G}$

There are blockwise versions $\mathsf{T}_{c,p,B}$ and $\Lambda_{c,p,B}$ of these, and

$$\Lambda_{c,p,B}(x) = \sum_{\chi \in \mathrm{Irr}(B)} |\chi(x)|^2$$

for all p-regular $x \in G$, while

$$\langle \Lambda_{c,p,B}, 1 \rangle = \ell(B).$$

The Lefschetz conjugation module $\mathcal{L}_c(B)$ is the element

$$\sum_{\sigma \in \mathcal{S}_p(G)/G} (-1)^{|\sigma|} \mathrm{Ind}_{G_\sigma}^{G}(B_\sigma)$$

of the Green ring of RG. The individual terms B_σ are not in general projective modules, but $\mathcal{L}_c(B)$ is a virtual projective. We also have

$$\mathcal{L}_c(B) = \sum_{\sigma \in \mathcal{S}_p(G)/G} (-1)^{|\sigma|} \mathrm{Ind}_{G_\sigma}^{G}(\mathsf{T}_{c,p,B_\sigma})$$

in the Green ring, and the individual terms $\mathsf{T}_{c,p,B_\sigma}$ are projective modules.

“Möbius inversion” (as in KR89, but with modules instead of integers) gives a (non-conjectural) identity:

$$\mathsf{T}_{c,p,B} = \sum_{Q \in \mathcal{Q}} \mathsf{Ind}_{N_G(Q)/Q}^{G} \left(P[\mathcal{L}_c(B(Q))] \right),$$

where $P[X]$ denotes the (virtual) projective cover of X as $RN_G(Q)$-module when X is a virtual projective $RN_G(Q)/Q$-module.

Hence the number of “fixed-points” on $\mathcal{L}_c(B(Q))$ should be the number of block of defect zero summands of $B(Q)$ for each Q. This gives a Green ring formulation of AWC, and I regard this as a “representation-theoretic” formula for $\mathsf{T}_{c,p,B}$ (and hence for $\mathsf{T}_{c,p,G}$).

The module $T_{c,p,G}$ also has a "group-theoretic" description: we have

$$\mathsf{T}_{c,p,G} = \sum_{y \in G_{p'}/G} \mathrm{Ind}_{C_G(y)}^{G}(P_{1,p,C_G(y)}),$$

where $P_{1,p,G}$ is the unique virtual projective RG-module whose virtual character $\Psi_{1,p,G}$ takes value $|G_p \cap C_G(x)|$ for each p-regular $x \in G$. A key property of $P_{1,p,G}$ is that it contains the projective cover of the trivial module with multiplicity one.

The virtual module $P_{1,p,G}$ (which we conjecture is always a genuine projective module) may be expressed in terms of the Steinberg module St_p as defined in the Green ring by P.J. Webb:

We have (for all finite groups G),

$$P_{1,p,G} = \sum_{Q \in \mathcal{Q}} \mathrm{Ind}_{N_G(Q)}^{G}(P[\mathrm{St}_p(N_G(Q)/Q)]),$$

where

$$\mathrm{St}_p(X) = \sum_{\sigma \in \mathcal{S}_p(X)/X} (-1)^{|\sigma|} \mathrm{Ind}_{X_\sigma}^{X}(R).$$

Here, $\mathcal{S}_p(X)$ is the p-subgroup complex of X and if $\sigma = Q_1 < Q_2 < \ldots < Q_r$, then $|\sigma| = r$.

Summary: Suppose that $O_p(G) = 1$. Then for each block B, the virtual module $T_{c,p,B} - \mathcal{L}_c(B)$ is determined by strictly p-local information. Also $P_{1,p,G} - \mathrm{St}_p(G)$ is determined by strictly p-local information. Furthermore, $T_{c,p,RG}$ is the sum of the modules induced from $P_{1,p,C_G(y)}$ as y runs over p-regular conjugacy class representatives.

Furthermore, Alperin's weight conjecture is equivalent to proving that the multiplicity of the projective cover of the trivial module in $\mathcal{L}_c(B)$ is one if B has defect zero, and is zero if B has positive defect.

Virtual modules in action: The case $p = 2$ and $G = \mathsf{SL}(2, 2^n)$

We consider this case for the purposes of illustration, to show that the theory for the general group G fits well here with the usual Lie-theoretic methodology. We explicitly determine the module $\mathsf{T}_{c,p,G}$, virtual modules

$$P_{x,2,G} = \mathrm{Ind}_{C_G(x)}^G(P_{1,2,C_G(x)}),$$

and (for each block B of RG), $\mathcal{L}_c(B)$.

Let $\sigma : \mathbb{F} \to \mathbb{F}$ be the Frobenius automorphism of $\mathbb{F}$ with $f\sigma = f^2$. Let S denote the (self-dual) natural module for G, realized over the field $\mathbb{F}_{2^n} \subseteq \mathbb{F}$.

Let $I = \{0, 1, 2, \ldots, n-1\}$. For each non-empty subset J of I, let

$$S_J = \otimes_{j \in J} S^{\sigma^j},$$

and let $S_\emptyset$ denote the trivial module $\mathbb{F}$. Then each S_J is simple, since $S_I = \mathrm{St}$ is simple. For each subset J of I, denote $I \setminus J$ by J'.

The projective cover of $S_\emptyset$ is $\mathrm{Ind}_{T_+}^G(\mathbb{F})$, where T_+ is a non-split torus of G of order 2^n+1, and for each non-empty proper subset J of I, the projective cover of S_J is $S_{J'} \otimes \mathsf{St}$.

We have

$$S_J \otimes P(S_J) \cong \mathsf{St} \otimes \mathsf{St}$$

for each non-empty subset J of I. We note also that $P(\mathbb{F})$ is a summand of $\mathsf{St} \otimes \mathsf{St}$, and we see that

$$\mathsf{St} \otimes \mathsf{St} \cong P(\mathbb{F}) \oplus \mathsf{St}.$$

Lifting to projective RG-modules, we have

$$\mathsf{T}_{c,2,G} = 2^n P(R) \oplus (2^n - 1)\mathsf{St}$$

in the Green ring (with obvious abuse of notation).

We recall that

$$\mathsf{T}_{c,2,G} = \sum_{x \in G_{2'}/G} P_{x,2,G}.$$

If x is a non-identity element of the non-split torus T_+ of order $2^n + 1$, then we have

$$P_{1,2,C_G(x)} = R$$

and $C_G(x) \cong T_+$. There are 2^{n-1} conjugacy classes of such elements x, and we obtain $P_{x,2,G} = P(R)$ for each such x.

If T_- is a split torus of G of order $2^n - 1$ and y is a non-identity element of T_-, then since there are $2^{n-1} - 1$ conjugacy classes of such y, each with $P_{1,2,C_G(y)} \cong R$, we find that

$$P_{y,2,G} \cong P(R) \oplus 2\mathrm{St}.$$

This yields

$$P_{1,2,G} = P(R) + \mathsf{St}.$$

We now have $P_{y,2,G} - P_{1,2,G} = \mathsf{St}$ for each non-identity $y \in T_-$ and $P_{1,2,G} - P_{x,2,G} = \mathsf{St}$ for each non-identity $x \in T_+$.

Let us examine the Lefschetz conjugation module $\mathcal{L}_c(RG)$. Here, the blocks of RG are the principal block B and the block B_s containing the Steinberg module St

Recall that we have

$$\mathsf{T}_{c,p,B} = \sum_{Q/G} \mathsf{Ind}_{N_G(Q)}^G (P[\mathcal{L}_c(B(Q))]),$$

but here, we only need to consider $Q = 1$ and $Q = U \in \mathsf{Syl}_2(G)$.

We know that $\mathcal{L}_c(B)$ is a virtual projective, and as $N_G(U)/U$ is a cyclic group of order $2^n - 1$, we see that $\mathcal{L}_c(B(U))$ is just the direct sum of $2^n - 1$ copies of the trivial module R (for $B(U) = RN_G(U)/U$ in this case). The projective cover of the trivial module of $N_G(U)$ is equal to $\mathrm{Ind}_{T_-}^{N_G(U)}(R)$.

We saw earlier that

$$\mathrm{Ind}_{T_-}^{G}(R) = 2\mathrm{St} + P(R).$$

Hence our formula simplifies in the Green ring to

$$T_{c,2,B} = [B - \mathrm{Ind}_{N_G(U)}^{G}(RN_G(U))]$$

$$+(2^n - 1)[2\mathrm{St} + P(R)].$$

Our module formula above now simplifies (after some rearrangement) in the Green ring to

$$B + 2^n\mathsf{St} = \mathsf{Ind}_{N_G(U}^G(RN_G(U)).$$

Hence in the Green ring for RG, we have

$$\mathcal{L}_c(B) = -2^n\mathsf{St}$$

and

$$\mathcal{L}_c(RG) = P(R) - (2^n - 1)\mathsf{St}.$$

Note in particular that $\mathcal{L}_c(B)$ contains the projective cover of the trivial module with multiplicity zero, and that

$$\mathcal{L}_c(B_s) = \mathsf{St} \otimes \mathsf{St} = P(R) + \mathsf{St}.$$